Last Time: Overview of our progress...

Fun Fact: If θ is any angle, then

$$M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$ is the transformation matrix of the map

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates every vector of $\mathbb{R}^2$ by θ radians counterclockwise $\left(\text{i.e. } M = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(R_\theta)\right)$.

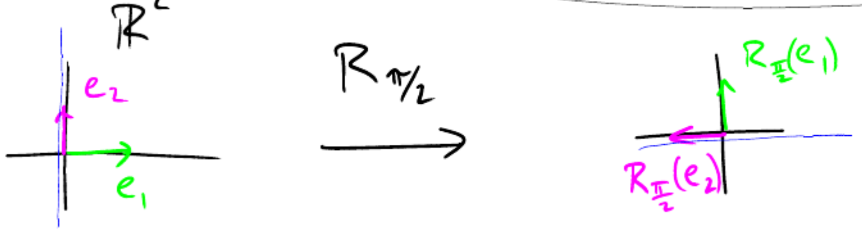
NB: Can be proved pretty easily... just check for all $0 \neq v \in \mathbb{R}^2$ that Mv is at angle θ with v...

Let $\theta = \frac{\pi}{2}$. Then $\cos(\theta) = 0$, $\sin(\theta) = 1$ so

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(R_{\frac{\pi}{2}}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Recall: If λ is an eigenvalue of operator $L : \mathbb{C}^n \to \mathbb{C}^n$ with algebraic mult α and geometric mult γ, then $\underbrace{1 \leq \gamma}_{\text{every e-value has at least 1 nonzero e-vector}} \leq \alpha$.

For $R_{\frac{\pi}{2}}$:



If $0 \neq v$ is an eigenvector of $R_{\frac{\pi}{2}}$, then $R_{\frac{\pi}{2}}(v) = \lambda v$ for some λ...

Q: Where is such a (nonzero) v in our picture?

A: There is none... $R_{\frac{\pi}{2}}$ has complex eigenvalues...

$$p_M(\lambda) = \det(M - \lambda I)$$
$$= \det\begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

So roots of $p_M(\lambda)$ (hence the eigenvalues of M)

$\lambda = \pm i$. Point: Eigenvectors of $R_{\frac{\pi}{2}}$ live in $\mathbb{C}^2$, not $\mathbb{R}^2$... Indeed:

$\lambda = i$: $M - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{i\ell_1} \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$-i \cdot i = -(i^2) = -(-1) = 1$

$\therefore$ System has homogeneous solutions $x - iy = 0$

i.e. $\begin{bmatrix} x \\ y \end{bmatrix} \in V_i$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y\begin{bmatrix} i \\ 1 \end{bmatrix}$.

$\therefore$ $V_i = \operatorname{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$, and ① alg mult of $\lambda = i$ is 1

② geom mult of $\lambda = i$ is 1.

$\boxtimes$

$\lambda = -i$: Do as an exercise...

Point: We really ought to think of our linear operators as operators on $\mathbb{C}^n$!!!

Diagonalizability

Defn: A square matrix M is diagonalizable when M is similar to a diagonal matrix.

(i.e. $M = P^{-1} D P$ for some P invertible and D diagonal).

Q: If M is diagonalizable, how do we diagonalize?

$$\begin{tikzcd}
V_B \arrow[r, "\mathrm{Rep}_{B,B}(L)=M"] \arrow[d, shift right, "\mathrm{Rep}_{B,E}(id)"'] & V_B \arrow[d, shift left, "\mathrm{Rep}_{B,E}(id)"] \arrow[d, shift right, "Q^{-1}"'] \\
V_E \arrow[r, "\mathrm{Rep}_{E,E}(L)=D"'] \arrow[u, shift right, bend right, "Q"'] & V_E
\end{tikzcd}$$

want to compute this! (E).

$$D = \mathrm{Rep}_{E,E}(L) = \mathrm{Rep}_{B,E}(id)\,\mathrm{Rep}_{B,B}(L)\,\mathrm{Rep}_{E,B}(id)$$

$$= Q^{-1} M Q$$

In particular, $QDQ^{-1} = (QQ^{-1})M(Q^{-1}Q)$
$= (I)M(I) = M$

So for $P^{-1} = Q$ (i.e. $P = Q^{-1}$), we see $M = P^{-1}DP$.

New Goal: Find a suitable basis E to replace B...

The diagonal matrix $D = \mathrm{Rep}_{E,E}(L)$ acts on elements of E as eigenvectors! If $E = \{v_1, v_2, \dots, v_n\}$ then $\mathrm{Rep}_E(v_i) = e_i$ ← standard basis vector...

So $\mathrm{Rep}_E(L(v_i)) = \mathrm{Rep}_{E,E}(L)\cdot \mathrm{Rep}_E(v_i) = De_i = d_{i,i}\, e_i$

where $D = [d_{i,j}]_{i,j=1}^{n,n} = \begin{bmatrix} d_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & d_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & d_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{n,n} \end{bmatrix}$

Point: $L(v_i) = d_{i,i}\, v_i$ so:

① v_i is an eigenvector of L. ② $d_{i,i}$ is the eigenvalue associated with v_i

③ E is actually a basis of V consisting entirely of eigenvectors of L!

Algorithm for Diagonalization: Let $M \in M_{n\times n}(\mathbb{C})$.

① Compute $p_M(\lambda) = \det(M - \lambda I)$ Characteristic polynomial of M.

② Compute the roots of $p_M(\lambda)$ (i.e. solve $p_M(\lambda) = 0$ to obtain the eigenvalues of M).

③ Compute the eigenspaces V_λ associated to each eigenvalue λ. (i.e. compute a basis B_λ of eigenvectors for each V_λ).

④ If $E = \bigcup_{\substack{\lambda \text{ an} \\ \text{e-value}}} B_\lambda$ $\underline{\text{is a basis of } \mathbb{C}^n}_{\text{geom mult = alg mult for all } \lambda \ldots}$, then we have computed the desired E. Otherwise, M is <u>not</u> diagonalizable!!!

Remarks: ① In steps 3 - 4, we used the fact that if $I \subseteq V_\lambda$ and $J \subseteq V_\mu$ are indep and $\lambda \neq \mu$, then $I \cup J$ is also indep in V.

Reason: $V_\lambda \cap V_\mu = \{0_V\} \leftarrow$ very easy :)

② As part of our construction of E, we noted the entries on the diagonal of D are the eigenvalues of M :)

Ex: Let $M = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$. We diagonalize M as follows:

Characteristic Poly:

$$p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 3-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)$$

Eigenvalues: $p_M(\lambda) = 0 \iff (3-\lambda)(1-\lambda) = 0 \iff 3-\lambda = 0$ OR $1-\lambda = 0$

$$\iff \lambda = 3 \quad \text{OR} \quad \lambda = 1$$

Eigenspaces:

$\underline{\lambda = 1}$: $M - I = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \text{RREF} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore V_1 = \text{null}(M - I) \ni \begin{bmatrix} x \\ y \end{bmatrix} \iff x + y = 0$

$\iff x = -y$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore B_1 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis of V_1

$\underline{\lambda = 3}$: $M - 3I = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \rightsquigarrow \text{RREF} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_3 = \text{null}(M - 3I) \iff y = 0$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\therefore B_3 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis of V_3.

Basis Change: Let $E = B_1 \cup B_3 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. E is a basis of $\mathbb{C}^2$ because $\mathbb{C}^2$ has dimension 2 and $\#E = 2$

$\text{Rep}_{E, \mathcal{E}_2}(\text{id}) = \text{Rep}_{\mathcal{E}_2, E}(\text{id})^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$ is computed via:

$$[E \mid \mathcal{E}_2] = \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] = [I \mid \underline{\text{Rep}_{E, \mathcal{E}_2}(\text{id})}]$$

$$\begin{tikzcd} V_E \arrow[r, "D"] \arrow[d, "\text{Rep}_{E,\mathcal{E}_2}(\text{id})"'] & V_E \\ V_{\mathcal{E}_2} \arrow[r, "M"'] & V_{\mathcal{E}_2} \arrow[u, "\text{Rep}_{\mathcal{E}_2,E}(\text{id})"'] \end{tikzcd}$$

$$D = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \quad \square$$